Generally try to classify series by its <u>form</u>.

1. <u>If</u> Series is of the form

$$\sum \frac{1}{n^p}$$

<u>THEN</u> converges for $p>1$
diverges for $p \le 1$

2. <u>If</u> has the form

$$\sum ar^{n-1} \quad \underline{\text{or}} \quad \sum ar^n$$

<u>THEN</u> geometric, so

converges for $|r|<1$ Limit is $\frac{a}{1-r}$
diverges for $|r| \geq 1$

3. <u>If</u> form is similar to geometric series or to p-series,
<u>then</u> try a comparison test.

a) <u>if</u> a_n is a rational function or algebaic function of n,
<u>then</u> compare w/ p-series (keeping only highest powers of $\underline{n}$ in numerator & denominator)

- If Σa_n has some negative terms, use comparison test ON $\Sigma |a_n|$ and test for absolute convergence.

4. <u>If</u> obvious that $\lim_{n\to\infty} a_n \neq 0$, use Test for Divergence.

5. $\Sigma (-1)^{n-1} b_n$, Alternating Series Test

6. Factorials or other products (including (constant)^n), try <u>Ratio TEST.</u>

7. <u>If</u> a_n has form $(b_n)^n$, try <u>ROOT</u> test

8. <u>If</u> $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easy & Conditions are satisfied,
<u>Then</u> try <u>Integral Test</u>

[10.1]

- Sequences
  1. Limit exists (not necessarily $L=0$) $\longrightarrow$ convergence
  2. L'HOPITAL'S RULE
  3. increasing, decreasing, monotonic

[10.2]

1. Geometric Series
2. Harmonic Series
3. Telescoping Series

[10.3]

1. Integral test
2. P-Series *

[10.4]

1. Comparison test
2. Limit Comparison test (when inequality in comparison test points wrong way)

[10.5]

1. Alternating Series test

[10.6]

1. Ratio Test
2. Root test

[A]

Practice a couple of Error bound computations

Thm: [10.1-2]

If $\lim_{x \to \infty} f(x) = L$ and $f(n) = a_n$ where $n \in \mathbb{Z}$,

then $\lim_{n \to \infty} a_n = L$

THM: [10.1] Squeeze theorem

If $a_n \le b_n \le c_n$ for $n \geq n_0$ and $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L$

then $\lim_{n \to \infty} b_n = L$

Thm: [10.1-5]

If $\lim_{n \to \infty} |a_n| = 0$

then $\lim_{n \to \infty} a_n = 0$

[10.1-7]

$$\lim_{n \to \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Thm [10.1-10]

Every bounded, monotonic sequence converges

[10.2]

2. Definition:

Given $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots$, let S_n denote its $\underline{n}^{th}$ partial sum:

$$S_n = \sum_{n=1}^{k} a_i = a_1 + a_2 + a_3 + \ldots + a_k$$

If the sequence $\{S_n\}$ is convergent and $\lim_{n \to \infty} S_n = S$ exist as a real number, the the series is called convergent and we write

$$a_1 + a_2 + a_3 + \ldots = S \quad \underline{\text{or}} \quad \sum_{a_k=1}^{\infty} a_k = S.$$

the number $\underline{S}$ is called the <u>sum</u> of the series. Otherwise we say the series is divergent.

4. The geometric series

NOT k, but (k-1)

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + \ldots$$

converges if $|r| < 1$ and its sum is

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}.$$

6. THM: If the series $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n \to \infty} a_n = 0$

[NOTE converse is <u>NOT</u> true in general.]

7. THM If $\sum a_n$ and $\sum b_n$ are convergent series, then so are

1. $\sum_{k=1}^{\infty} c a_k = c \sum_{k=1}^{\infty} a_k$

2. $\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$

3. $\sum_{k=1}^{\infty} (a_k - b_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$

10.3 The integral test AND estimates of Sums

[2]

The Integral test:

Suppose f is continuous, positive, decreasing on $[1, \infty]$

then $\sum_{n=1}^{\infty} a_n$ converges iff $\int_1^{\infty} f(x)\,dx$ converges.

[3] P-Series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$, and diverges otherwise

[4] Remainder Estimate for the integral test

If $\sum a_n$ converges by the integral test

and $R_n = S - S_n$

then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$$

[5]

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \leq S \leq S_n + \int_n^{\infty} f(x)\,dx$$

[10.4] COMPARISON TESTS

1. Compare the given series to a series whose convergence behavior is already known:

Suppose Σa_n And Σb_n are series w/ positive terms.

a) if Σb_n converges <u>and</u> $a_i \le b_i$ for all i,
then a_n converges too.

b) <u>if</u> Σb_n diverges <u>and</u> $a_i \ge b_i$ for all i,
then Σa_n diverges too.

2. The Limit Comparison TEST

Spose $a_k, b_k > 0$

Let $L = \lim\limits_{k \to \infty} \frac{a_k}{b_k}$

3-CASES:

a) <u>IF</u> $0 < L < \infty$ <u>THEN</u> Either $\sum\limits_{k=1}^{\infty} a_k$ and $\sum\limits_{k=1}^{\infty} b_k$ both diverge
OR they both converge.

b) <u>If</u> $L = 0$ <u>AND</u> $\sum\limits_{n=1}^{\infty} b_k$ converges <u>THEN</u> $\sum\limits_{k=1}^{\infty} a_k$ Converges too.

c) <u>IF</u> $L = \infty$ <u>AND</u> $\sum\limits_{n=1}^{\infty} b_k$ diverges <u>THEN</u> $\sum\limits_{k=1}^{\infty} a_k$ Diverges too.

[10.5]

1. THE ALTERNATING SERIES TEST

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \dots$$

Satisfies

a) $b_k > 0$

and b) $b_{k+1} \leq b_k$ for all k

and c) $\lim_{k\to\infty} b_k = 0$

THEN the series CONVERGES.

[NB: The Alternating Harmonic Series CONVERGES]

1a) Estimating Sums

If $\sum_{k=1}^{\infty}(-1)^{k-1}b_k$ converges by AST

then

magnetude of $n^{th}+1$ term

$$|R_n| = |s - s_n| \leq b_{n+1}$$

error associated w/ nth partial sum

[10.6] Absolute Convergence
Ratio test
Root test

1. Def — Given a series $\sum a_k$,
If the series of absolute values $\sum |a_k|$ converges,
then we say the series is absolutely convergent.

2. Def — A series is called conditionally convergent if it is convergent but not absolutely convergent.

3. THM — **If** a series $\sum a_k$ is absolutely convergent,
then it is convergent.

4. THE RATIO TEST

a) **If** $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = L < 1$,
then the series $\sum a_k$ is absolutely convergent

b) **If** $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = L > 1$ OR $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \infty$,
then, the series $\sum a_k$ is divergent,

5. The ROOT TEST

a) **If** $\lim_{k\to\infty} \sqrt[k]{|a_k|} = L < 1$, **then** $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

b) **If** $\lim_{k\to\infty} \sqrt[k]{|a_k|} = L > 1$ or $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \infty$, then $\sum_{k=1}^{\infty}$ diverges.